EQUIVALENCES FOR MORSE HOMOLOGY

MATTHIAS SCHWARZ

Abstract. An explicit isomorphism between Morse homology and singular homology is constructed via the technique of pseudo-cycles. Given a Morse cycle as a formal sum of critical points of a Morse function, the unstable manifolds for the negative gradient flow are compactified in a suitable way, such that gluing them appropriately leads to a pseudo-cycle and a well-defined integral homology class in singular homology.

1. Introduction

The aim of this paper is to give an explicit construction of an isomorphism between Morse homology and singular homology. Morse homology is a Morse-theoretical approach to the homology of a smooth manifold which goes back already to Thom and plays a crucial role in Smale's proof of the h-cobordism theorem, cf. also [Mil65]. It was studied by J. Franks [Fra79], rediscovered by Witten [Wit82] in terms of a deformation of the de Rham complex and generalized by Floer [Flo89] as an approach to solve a conjecture by Arnold. In [Sch93] the author developed a comprehensive approach to Morse homology as an axiomatic homology theory for the category smooth manifolds (not necessarily compact) satisfying all Eilenberg-Steenrod axioms. Moreover, this approach used the purely relative "Floer-theoretical" definition of Morse homology in terms of moduli spaces of trajectories for the gradient flow equation connecting critical points. However, [Sch93] did not present an explicit isomorphism to other axiomatic homology theories like for instance the de Rham theorem between de Rham cohomology and singular cohomology. That Morse homology is isomorphic to other homology theories is proved in [Sch93] by extending it to a slightly larger category of certain CW-spaces compatible with the manifold structure in which the isomorphism is deduced inductively, based on the Eilenberg-Steenrod axioms. That is, in such a category existence and uniqueness of the isomorphism follows by abstract application of the axioms.

In the approach of Smale and Milnor, used similarly also in Floer's description, a direct isomorphism between Morse homology and singular homology is obtained by choosing a special, namely self-indexing Morse function, such that the boundary map in the Morse chain complex can be related to the connecting homomorphism ∂_* in the long exact sequence of the cell decomposition induced by the Morse function (see also Section 4.1.1 below).

The approach of this paper is to show that, given *any* Morse function f with a generic Riemannian metric g, one can construct singular cycles explicitly from the given Morse cycle. The main objects which have to be considered as intermediate tools are so-called *pseudo-cycles*. This is a geometric differential-topological way

Date: March 9. 1999.

1991 *Mathematics Subject Classification.* Primary 58E05; Secondary 55N35, 57R70.

The author was supported in part by NSF Grant # DMS 9626430.

to represent homological (integral!) cycles which plays also a role in the definition of quantum cohomology (see e.g. [MS94]). In this paper, we give a short proof that integral homology classes can be represented by pseudo-cycles and that every pseudo-cycle in fact leads to an integral homology class.

The purpose of this paper is also to provide a detailed construction of this equivalence between Morse homology and singular homology via pseudo-cycles, which to the author's knowledge has not yet been carried out elsewhere, but which already has been used several times, in particular in the theory of quantum cohomology and Floer homology, e.g. [PSS96], [Sch98], [Sei97], [Sch].

After a short account on the definition of Morse homology pseudo-cycles are defined in Section 3, where it is proven that pseudo-cycles represent integral homology classes and every class can be represented as such. Section 4 contains the construction of the explicit isomorphism between Morse homology and singular homology. In the first part we show how to obtain a well-defined pseudo-cycle from a given Morse cycle and that the induced singular class is uniquely associated to the Morse homology class. The idea is to glue all unstable manifolds of critical points, which occur in the given Morse cycle, along the 1-codimensional strata of their suitable compactifications. In the second part we construct the inverse homomorphism in terms of intersections of pseudo-cycles representing singular classes and stable manifolds of critical points.

2. Morse homology

2.1. **Definition.** Let M be an oriented[1] smooth manifold, $f \in C^\infty(M,\mathbb{R})$ an exhausting[2] Morse function and g be a complete Riemannian metric. Consider the critical set $\mathrm{Crit}_* f$ of f as graded by the Morse index $\mu\colon \mathrm{Crit}\, f \to \mathbb{Z}$ and define the stable and unstable manifolds of the negative gradient flow in terms of spaces of curves,

$$\begin{aligned} W^u(x) &= \{\, \gamma\colon (-\infty,0] \to M \mid \dot\gamma + \nabla_g f\circ\gamma = 0,\ \gamma(-\infty) = x \,\}, \\ W^s(y) &= \{\, \gamma\colon [0,\infty) \to M \mid \dot\gamma + \nabla_g f\circ\gamma = 0,\ \gamma(+\infty) = y \,\} \end{aligned} \tag{2.1}$$

for $x, y \in \mathrm{Crit}\, f$. The curves γ are smooth and $\gamma(\pm\infty)$ denotes the limit for $t \to \pm\infty$. The spaces $W^u(x)$ and $W^s(y)$ are finite-dimensional manifolds with

$$\dim W^u(x) = \mu(x) \quad \text{and} \quad \dim W^s(y) = \dim M - \mu(y)$$

and the evaluation mapping $\gamma \to \gamma(0)$ induces smooth embeddings into M, i.e. diffeomorphisms onto the image,

$$E_x\colon W^u(x) \hookrightarrow M, \quad E_y\colon W^s(y) \hookrightarrow M\,.$$

However, in general, these maps are not proper. Choosing a generic Riemannian metric we obtain Morse-Smale transversality, namely $W^u(x)$ and $W^s(y)$ intersect transversely in M with respect to E_x and E_y. If this transversality holds for all $x, y \in \mathrm{Crit}\, f$, (f, g) is called a *Morse-Smale* pair. We obtain the manifold of connecting orbits

$$\begin{aligned} M_{x,y}(f,g) &= W^u(x) \pitchfork W^s(y) \\ &= \{\, \gamma\colon \mathbb{R} \to M \mid \dot\gamma + \nabla_g f\circ\gamma = 0,\ \gamma(-\infty) = x,\ \gamma(+\infty) = y \,\}, \\ \dim M_{x,y}(f,g) &= \mu(x) - \mu(y), \end{aligned}$$

[1]If M is not orientable, choose homology coefficients in $\mathbb{Z}_2$.

[2]i.e. proper and bounded below. In [Sch93], this property is called coerciveness.

on which, if $x \neq y$, $\mathbb{R}$ acts freely and properly by shifting

$$(\tau * \gamma)(t) = \gamma(t+\tau)\,.$$

Let us fix orientations for all unstable manifolds $W^u(x)$, then the orientation of M induces orientations for $W^s(y)$ and $M_{x,y}$. We call an unparameterized trajectory $\hat\gamma \in M_{x,y}/\mathbb{R}$ for relative index 1 positively oriented if the orbit $\mathbb{R}\cdot\hat\gamma \subset M_{x,y}$ is positively oriented by the action of $\mathbb{R}$ which corresponds to the action by the negative gradient flow. Thus, for relative index 1, the moduli spaces of unparameterized trajectories

$$\widehat{M}_{x,y} = M_{x,y}/\mathbb{R}, \quad \mu(x)-\mu(y)=1,$$

are compact, that is finite, and every element $\hat\gamma$ carries a sign $\tau(\hat\gamma) \in \{\pm 1\}$. We define the intersection numbers

$$n(x,y) = \sum_{\hat\gamma \in \widehat{M}_{x,y}} \tau(\hat\gamma)$$

and an operator on the module over $\mathbb{Z}$ generated by the critical points of index k,

$$C_k(f) = \mathbb{Z}\otimes \mathrm{Crit}_k f, \quad \partial = \partial(f,g),$$
$$\partial\colon C_k(f) \to C_{k-1}(f), \quad \partial x = \sum_y n(x,y)\,y\,.$$

The fundamental theorem of Morse homology is

Theorem 2.1. *∂ is a chain boundary operator, i.e. $\partial\circ\partial = 0$.*

Hence, the homology $H_k(f,g;\mathbb{Z}) = H_k(C_*(f), \partial(f,g))$ is well-defined as the quotient of the module of *Morse-cycles*

$$Z_k(f,g) = \{\, a = \sum_{x\in \mathrm{Crit}_k f} a_x x \mid \partial a = 0\,\}\,.$$

modulo the boundaries $B_k(f,g) = \operatorname{im}\partial$.

Let us now recall the homotopy invariance result in Morse homology. It is based on Conley's continuation principle (see [Con78]).

Theorem 2.2. *Given two Morse-Smale pairs (f^0,g^0) and (f^1,g^1) there exists a canonical homomorphism*

$$\Phi_{10}\colon H_*(f^0,g^0) \to H_*(f^1,g^1)$$

such that

$$\Phi_{21}\circ\Phi_{10} = \Phi_{20} \quad \textit{and} \quad \Phi_{00} = \mathrm{id}\,.$$

In particular, every Φ_{ji} is an isomorphism.

This continuation theorem implies that we have well-defined Morse homology groups

$$\begin{aligned} H_*^{\mathrm{Morse}}(M;\mathbb{Z}) &\overset{\mathsf{def}}{=} \{\, (a_i) \in \prod H_*(f^i,g^i) \mid a_j = \Phi_{ji} a_i \,\} \\ \rho_i\colon H_*(f^i,g^i) &\xrightarrow{\cong} H_*^{\mathrm{Morse}}(M;\mathbb{Z}), \quad \rho_i\circ\Phi_{ij} = \rho_j\,. \end{aligned} \tag{2.2}$$

Let us recall the construction of Φ_{10} from [Sch93]. Given the Morse-Smale pairs (f^i,g^i), $i=0,1$, we choose an asymptotically constant homotopy over $\mathbb{R}$, (f_s,g_s), $s\in\mathbb{R}$ with

$$(f_s,g_s) = \begin{cases} (f^0,g^0), & s \leqslant -R, \\ (f^1,g^1), & s \geqslant R, \end{cases}$$

for R large enough. This gives rise to the trajectory spaces

$$M_{x_o,x_1}(f_s,g_s) = \{\, \gamma \mid \dot\gamma(s) + \nabla_{g_s} f_s(\gamma(s)) = 0,\\ \gamma(-\infty) = x_o,\ \gamma(\infty) = x_1 \,\}\,.$$

For a generic choice of the homotopy (f_s, g_s), these spaces are finite dimensional manifolds with

$$\dim M_{x_o,x_1} = \mu(x_o) - \mu(x_1)$$

and compact in dimension 0. As in the definition of the boundary operator ∂ we define

$$\Phi_{10}\colon C_*(f^0,g^0) \to C_*(f^1,g^1),$$
$$\Phi_{10} x_o = \sum_{x_1} n(x_o,x_1) x_1,$$
$$n(x_o,x_1) = \#_{\mathrm{alg}} M_{x_o,x_1}(f_s,g_s)\,,$$

where $\#_{\mathrm{alg}}$ means counting with signs $\tau(u) = \pm 1$, analogously to above. That Φ_{10} is well-defined on the level of homology follows from a theorem stating that

$$\Phi_{10} \circ \partial_0 = \partial_1 \circ \Phi_{10}\,.$$

Moreover, it is shown in [Sch93] that the homomorphism Φ_{10} on homology level does not depend on the choice of the homotopy (f_s, g_s).

2.2. **Isomorphism via axiomatic approach.** In [Sch93], Morse homology is extended towards an axiomatic homology theory for the category of smooth manifolds. It is functorial with respect to smooth maps, there exists a relative version so that we have an associated long exact sequence, and all axioms of Eilenberg and Steenrod are satisfied. However, in order to derive a natural isomorphism with any other axiomatic homology theory, an extension to a larger category of spaces is required, e.g. towards the subcategory of CW-pairs which are embedded smoothly into finite-dimensional manifolds as strong deformation retracts of open subsets. This approach is adopted in [Sch93] in order to prove the equivalence with other homology theories.

3. Pseudo-cycle homology

In [MS94], pseudo-cycles were defined in order to find a suitable differential-topological representation of homology cycles. However, this was only used with rational coefficients so that every cycle can be represented as a closed submanifold. Here, we consider integral homology classes.

Let M be a compact[3] n-dimensional manifold. We consider an oriented smooth k-dimensional manifold without boundary V together with a smooth map $f\colon V \to M$. Let the set $f(V^\infty)$ be defined as in [MS94],

$$f(V^\infty) \stackrel{\mathsf{def}}{=} \bigcap_{K \subset V \text{ cpt.}} \overline{f(V \setminus K)}\,. \tag{3.1}$$

According to [MS94], $f\colon V \to M$ is a **pseudo-cycle** if $f(V^\infty)$ can be covered by the image of a smooth map $g\colon P \to M$ which is defined on a manifold P of dimension not larger than $\dim V - 2$.

[3]This poses no restriction for our application to Morse homology because we consider only cycles lying in the compact sublevel sets $M^a = \{\, p \in M \mid f(p) \leqslant a \,\}$ of an exhausting function.

Moreover, let W be an oriented smooth $(k+1)$-dimensional manifold with boundary ∂W, such that the inclusion $i\colon \partial W \hookrightarrow W$ is proper, and let $F\colon W \to M$ be a smooth map.

Theorem 3.1. *Let (f,V) be a pseudo-cycle and (F,W) as above.*

(a) *If $H_k(f(V^\infty);\mathbb{Z}) = H_{k-1}(f(V^\infty);\mathbb{Z}) = 0$ and $f(V) \not\subseteq f(V^\infty)$, then (f,V) induces a unique integral homology class $\alpha_f \in H_k(M;\mathbb{Z})$.*

(b) *Let $\partial W = U$ be an open subset of V such that $f(V^\infty) \subseteq f(U^\infty)$ and $F(W^\infty) \cap f(U) = \emptyset$. If $H_k(F(W^\infty);\mathbb{Z}) = 0$ the homology class α_f vanishes.*

In view of part (b) let us consider two pseudo-cycles $f_1\colon V_1 \to M$ and $f_2\colon V_2 \to M$ to be **cobordant** if their disjoint union $V = V_1 \amalg V_2^*$ with orientation on V_2 reversed forms a pseudo-cycle $f\colon V \to M$ such that there exists $F\colon W \to M$ satisfying the condition in (b).

In this section we are using Alexander-Spanier homology theory for locally compact Hausdorff spaces, cf. [Mas78]. The homology theory with arbitrary supports[4], i.e. not necessarily compact supports, is denoted by $H_*^\infty(X)$. Note that, however, homology theory with arbitrary supports, which is functorial with respect to proper maps, agrees with any homology theory with compact supports, as for instance singular homology, when restricted to compact sets as M, $f(V^\infty)$ and $F(W^\infty)$.

Proof of Theorem 3.1. Every oriented k-dimensional manifold X without boundary carries a uniquely defined fundamental class $[X] \in H_k^\infty(X;\mathbb{Z})$. If X is a manifold with boundary ∂X then $[X]$ is well-defined in $H_k^\infty(X \setminus \partial X) = H_k^\infty(X, \partial X)$. Every open subset $U \subset X$ inherits an orientation from X so that the natural restriction map $\rho\colon H_k^\infty(M;\mathbb{Z}) \to H_k^\infty(U;\mathbb{Z})$ gives $\rho([U]) = [X]$. Without loss of generality we may assume that

$$f(V) \cap f(V^\infty) = \emptyset\,.$$

Otherwise, we replace V by the open, nonempty subset $V \setminus f^{-1}(f(V^\infty))$. Since Alexander-Spanier homology with arbitrary supports is functorial with respect to proper maps of locally compact Hausdorff spaces we redefine the map

$$f\colon V \to M \setminus f(V^\infty)\,.$$

By definition of $f(V^\infty)$, f is proper. The integral class

$$(f)_*([V]) \in H_k^\infty(M \setminus f(V^\infty);\mathbb{Z})$$

is well-defined. From the exact homology sequence for H_*^∞ and the pair $(M, f(V^\infty))$,

$$H_k(f(V^\infty)) \to H_k(M) \xrightarrow{j_*} H_k^\infty(M \setminus f(V^\infty)) \to H_{k-1}(f(V^\infty))\,,$$

we obtain by assumption the isomorphism j_*. Hence,

$$\alpha_f \equiv j_*^{-1}(f)_*([V]) \in H_k(M;\mathbb{Z})$$

is well-defined.

We consider now an open subset $U \subset V$ such that $f(V^\infty) \subset f(U^\infty)$ and $f(U) \cap f(U^\infty) = \emptyset$. We carry out the same procedure as before for the proper map

$$f_U\colon U \to M \setminus f(U^\infty)$$

[4]Alternatively, we could also use Borel-Moore homology with specified type of supports.

and relate it to α_f by the following commutative diagram with respect to the natural restriction homomorphism ρ,

$$\begin{tikzcd}
H_k^\infty(V) \arrow[r, "f_*"] \arrow[d, "\rho"] & H_k^\infty(M \setminus f(V^\infty)) \arrow[d, "\rho"] \\
H_k^\infty(U) \arrow[r, "(f_U)_*"] & H_k^\infty(M \setminus f(U^\infty)) \,.
\end{tikzcd}$$

Since $\rho \circ j_* = j_*^U$ it follows that

$$j_*^U(\alpha_f) = \rho \circ f_*([V]) = (f_U)_*([U])\,. \tag{3.2}$$

Let us consider now the bordism $\partial W = U$. Without loss of generality we can assume that $F(W^\infty) \cap F(W) = \emptyset$ so that we have the proper map

$$F\colon W \to M \setminus F(W^\infty)\,.$$

In Alexander-Spanier homology theory, we know that the fundamental class $[U]$ is the image of the fundamental class of the manifold with boundary W under the boundary homomorphism ∂_* in the exact homology sequence of the pair (W, U). We obtain the commutative diagram

$$\begin{tikzcd}
H_{k+1}^\infty(W,U) \arrow[r, "\partial_*"] & H_k^\infty(U) \arrow[r, "i_*"] \arrow[d, "(f_U)_*"] & H_k^\infty(W) \arrow[d, "F_*"] \\
 & H_k^\infty(M \setminus f(U^\infty)) \arrow[r, "\rho"] & H_k^\infty(M \setminus F(W^\infty)) \\
 & H_k(M) \arrow[r, "\mathrm{id}"] \arrow[u, "j_*^U"] & H_k(M)\,, \arrow[u, "j_*^W"]
\end{tikzcd}$$

therefore by (3.2)

$$j_*^W(\alpha_f) = \rho \circ j_*^U(\alpha_f) = F_* \circ i_*([U]) = 0,$$

because $[U] \in \operatorname{im} \partial_*$. Since j_*^W is injective due to $H_k^\infty(F(W^\infty)) = 0$ it follows that α_f vanishes. $\square$

A topological space S is said to have **covering dimension** at most n if every open cover $\mathfrak{U} = \{U_\alpha\}$ has a refinement $\mathfrak{U}' = \{U'_\alpha\}$ for which all the $(n+2)$-fold intersections are empty[5],

$$U'_B = \bigcap_{\beta \in B} U'_\beta = \emptyset \quad \text{if } |B| \geqslant n+2\,.$$

We say then $\dim_{\mathrm{cov}} S \leqslant n$. Clearly, S having covering dimension at most n implies that $\check{H}^m(S) = 0$ for $m > n$. For a compact space S this implies $H_m(S) = 0$ for $m > n$.

We have the following simple

Lemma 3.2. *Let $f\colon P \to M$ be a smooth map between manifolds and S be a compact subset of M such that $S \subset f(P)$. Then*

$$\dim_{\mathrm{cov}} S \leqslant \dim P\,.$$

The final result is

[5] compare [DK90], Section 9.2.3.

Theorem 3.3. *Every pseudo-cycle $f\colon V \to M$ of dimension k induces a well-defined integral homology class $\alpha_f \in H_k(M;\mathbb{Z})$. Moreover, any singular cycle $\alpha \in Z_k^{\mathsf{sing}}(M;\mathbb{Z})$ gives rise to a k-pseudo-cycle $f\colon V \to M$ such that $\alpha_f = \alpha$.*

Note that if $f(V) \subset f(V^\infty)$ then trivially $\alpha_f = 0$.

Proof. The well-defined homology class α_f follows from combining Theorem 3.1 with Lemma 3.2.

Suppose now that $\alpha \in H_k^{\mathsf{sing}}(M;\mathbb{Z})$ is a k-cycle given by a smooth singular chain. By pairwise identifying and sufficiently smoothing the $k-1$-dimensional faces of the k-simplexes involved in α the cycle-property of α implies that we obtain a k-dimensional manifold V, not necessarily compact, with a smooth structure, such that the singular chain gives a map $f\colon V \to M$ meeting the pseudo-cycle condition, since $f(V^\infty)$ is covered by the images of the faces of codimension 2 and higher. Two cohomologous singular chains lead to cobordant pseudo-cycles in the sense of Theorem 3.1 (b). □

Hence, from now on we can represent integral cycles in singular homology by pseudo-cycles.

4. The Explicit Isomorphism

The first part consists of showing that each Morse cycle leads to a well-defined pseudo-cycle, and that the associated singular homology class does not depend on any uncanonical choices involved.

4.1. Pseudo-cycles from Morse cycles.

Let (f,g) be a Morse-Smale pair and consider the associated homology $H_*(f,g)$. The idea of defining the homomorphism into singular homology is to construct a k-dimensional pseudo-cycle $E\colon Z(a) \to M$ for a given Morse cycle $\{a\} = \{\sum_{x\in \mathrm{Crit}_k f} a_x\, x\ \in\} \in H_k(f,g)$. This is essentially based on considering the unstable manifolds $W^u(x)$ from (2.1) with multiplicity $a_x \in \mathbb{Z}$ and their evaluation maps $E_x\colon W^u(x) \to M$. In order to obtain a well-defined pseudo-cycle we have to carry out a suitable identification on the 1-codimensional strata of a suitable compactification of $W^u(x)$.

Let $x \in \mathrm{Crit}_k f$ and $y \in \mathrm{Crit}_{k-1} f$ such that $\widehat{M}_{x,y}$ is a nonempty finite set. We say that a sequence $(w_n) \subset W^u(x)$ is *weakly convergent* towards a simply broken trajectory,

$$w_n \rightharpoonup (\hat{u}, v) \in \widehat{M}_{x,y} \times W^u(y),$$

if $w_n \to v$ in $C^\infty_{\mathrm{loc}}((-\infty,0],M)$ and there exists a reparametrization sequence $\tau_n \to -\infty$ such that $\tau_n * w_n \to u$ in C^∞ on compact subsets of $\mathbb{R}$ for a representative u of the unparameterized trajectory $\hat{u}$. Note that, in particular, $w_n(0) \to v(0)$.

The following result is completely analogous to the gluing results developed in [Sch93], Section 2.5. There, the gluing operation has been constructed for trajectory spaces $M_{y,z}$ instead of $W^u(y)$, but the case of unstable manifolds is handled exactly the same. It provides us with the suitable description of strata of the weak compactification of $W^u(x)$.

Lemma 4.1 ([Sch93]). *Given an open subset $V \subset W^u(y)$ with compact closure there exists a constant $\rho_V > 0$ and a smooth map*

$$\begin{aligned}\#^V : \widehat{M}_{x,y} \times V \times [\rho_V, \infty) &\to W^u(x),\\ (\hat{u}, v, \rho) &\mapsto \hat{u}\#_\rho v,\end{aligned}$$

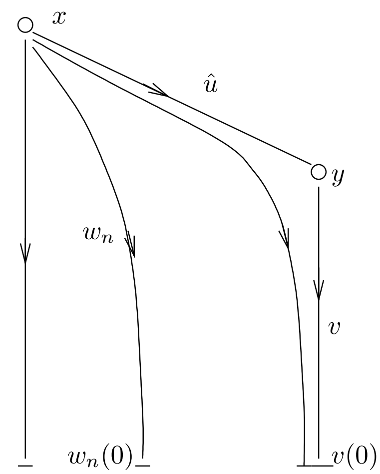

FIGURE 1. Weak convergence towards simply broken trajectory

such that

(a) $\#^V$ *is an embedding,*
(b) $\#^V(\hat{u},\cdot,\cdot)$ *is orientation preserving exactly if* $\tau(\hat{u})=+1$,
(c) $\hat{u}\#_\rho v \rightharpoonup (\hat{u},v)$ *for* $\rho\to\infty$, *and for any* $w_n \rightharpoonup (\hat{u},v)$ *there exists an* n_o *such that for all* $n\geqslant n_o$ $w_n=\hat{u}_n\#_{\rho_n}v_n$ *for unique* $(\hat{u}_n,v_n,\rho_n)$, *and*
(d) *the evaluation maps* $E_x\colon W^u(x)\to M$ *and* $E_y\colon W^u(y)\to M$ *extend to*

$$\bar{E}_x\colon W^u(x)\cup_{\#^V}\widehat{M}_{x,y}\times V\times[\rho_V,\infty)\to M\,.$$

such that $\bar{E}_x(\hat{u},v,\rho)=E_x(\hat{u}\#_\rho v)$ *for* $\rho\in[\rho_v,\infty)$ *and* $\bar{E}_x(\hat{u},v,\rho)=E_y(v)$ *for* $\rho=\infty$.

Let us define $\overline{W}^u(x)$ to be the disjoint union

$$\overline{W}^u(x)=W^u(x)\cup\bigcup_{\mu(y)=\mu(x)-1}\widehat{M}_{x,y}\times W^u(y) \tag{4.1}$$

equipped with the topology generated by

(a) the open subsets of $W^u(x)$,
(b) the neighborhoods of $(\hat{u},v)\in\widehat{M}_{x,y}\times W^u(y)$ of the form

$$\#^V(\{\hat{u}\}\times V\times(\rho,\infty))\cup\{\hat{u}\}\times V,\quad \rho\geqslant\rho_V\,,$$

for $V\subset W^u(y)$ open with compact closure.

This provides a Hausdorff topology and we obtain

Lemma 4.2. $\overline{W}^u(x)$ *is an oriented manifold with boundary oriented by* $\widehat{M}_{x,y}\times W^u(y)$ *and* $\bar{E}_x\colon\overline{W}^u(x)\to M$ *is a smooth embedding.*

The proof is given below together with the proof of Lemma 4.4.

Consider now a Morse-cycle

$$a\in Z_k(f,g),\quad a=\sum_x a_x x\,.$$

Given $l \in \mathbb{N}$ let us denote by $l \cdot \overline{W}^u(x)$ the disjoint union of l copies of $\overline{W}^u(x)$, that is, the topological sum. If $l \in \mathbb{Z}$, $l < 0$, we replace $\overline{W}^u(x)$ by $\overline{W}^u(x)^*$, that is, with the orientation reversed. Thus, we associate to a the topological sum

$$a \mapsto \amalg_x a_x \cdot \overline{W}^u(x) \tag{4.2}$$

which is a k-dimensional oriented manifold with oriented boundary and it consists of $\sum_x |a_x|$ connected components. Observe that this manifold with boundary is not compact in general.

We denote by Δa the following finite set of connecting unparametrized trajectories of relative index 1,

$$\Delta a = \bigcup \{\, a_x \hat{u} \mid \hat{u} \in \widehat{M}_{x,y},\ x \in \mathrm{Crit}_k f,\ y \in \mathrm{Crit}_{k-1} f \,\}$$

where $a_x \hat{u}$ is the disjoint union of $|a_x|$ copies of $\{\hat{u}\}$. Each $\hat{u}$ carries the sign $\tau(\hat{u}) \in \{\pm 1\}$ and we assign to every $\gamma \in a_x \hat{u}$ the new sign $\sigma(\gamma) = \mathrm{sgn}(a_x) \cdot \tau(\hat{u})$. Computing

$$\partial a = \sum_x \sum_{\mu(y)=\mu(x)-1} \sum_{\hat{u} \in \widehat{M}_{x,y}} a_x \tau(\hat{u})\, y$$

we immediately obtain

Lemma 4.3. *If $a = \sum_x a_x x$ is a Morse-cycle there exists an equivalence relation $\sim_{\Delta a}$ on Δa such that for each $\gamma \in \Delta a$ there exists a unique $\gamma' \neq \gamma$ with $\sigma(\gamma') = -\sigma(\gamma)$, so that $\gamma \sim_{\Delta a} \gamma'$ and $\gamma(+\infty) = \gamma'(+\infty) \in \mathrm{Crit}_{k-1} f$ for γ, γ' viewed as flow trajectories.*

Since Δa is an index set for the components of the $k-1$-dimensional manifold from (4.2), $\partial(\amalg_x a_x \overline{W}^u(x))$ such that $\sigma(\gamma)$ corresponds to the boundary orientation, we obtain the equivalence relation for points $\{\gamma\} \times \{v\} \in a_x \widehat{M}_{x,y} \times W^u(y)$,

$$\{\gamma\} \times \{v\} \sim_a \{\gamma'\} \times \{v'\} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \gamma \sim_{\Delta a} \gamma',\ v = v' \,.$$

We define

$$Z(a) = \amalg_x a_x \overline{W}^u(x) / \sim_a \,. \tag{4.3}$$

One easily sees that $Z(a)$ is a topological Hausdorff space and clearly the evaluation

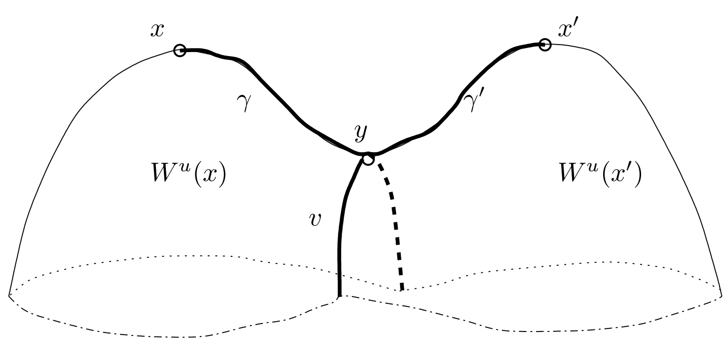

FIGURE 2. Gluing unstable manifold along simply broken trajectories

maps $\bar{E}_x$ yields

$$E\colon Z(a) \to M, \quad [\gamma, v] \mapsto v(0)\,.$$

In fact, we obtain

Lemma 4.4. *The space $Z(a)$ carries the structure of a k-dimensional manifold without boundary and $E\colon Z(a) \to M$ is a smooth map.*

Proof of Lemmata 4.2 and 4.4. Let us consider $x, x' \in \mathrm{Crit}_k f$ and $y \in \mathrm{Crit}_{k-1} f$ with $\hat{u} \in \widehat{M}_{x,y}$ and $\hat{u}' \in \widehat{M}_{x',y}$ such that $\hat{u} \sim_{\Delta a} \hat{u}'$, in particular, $\tau(\hat{u}) = -\tau(\hat{u}')$. Let $v_o \in W^u(y)$ and $V, V' \subset W^u(y)$ be two relatively compact neighborhoods of v_o.

In view of Lemma 4.2 we consider the following local coordinates at $(\hat{u}, v_o) \in \partial\overline{W}^u(x)$, respectively for Lemma 4.4 $[(\hat{u}, v_o)]_{\sim_a} \in Z(a) = \amalg_x a_x \overline{W}^u(x)/\sim_a$,

$$\#^V_{\hat{u},\hat{u}'}\colon V \times [0,\epsilon) \to \overline{W}^u(x),$$
$$(v,t) \mapsto \begin{cases} \hat{u}\#_{-\frac{1}{t}} v, & t < 0,\\ (\hat{u}, v_o) \sim_a (\hat{u}', v_o), & t = 0,\\ \hat{u}'\#_{\frac{1}{t}} v, & t > 0,\end{cases}$$

where $\#$ is the gluing map from Proposition 4.1 and $\epsilon > 0$ is small enough depending on the compact set $\mathrm{cl}(V)$. Thus, we have to show that

(A) $(\#^{V'}_{\hat{u},\hat{u}'})^{-1} \circ \#^V_{\hat{u},\hat{u}'}\colon U \times (-\epsilon_o, \epsilon_o) \to (V \cap V') \times \mathbb{R}$ is smooth for $U \subset V \cap V'$ and $\epsilon_o < \min(\epsilon, \epsilon')$ sufficiently small, and that

(B) $E \circ \#^V_{\hat{u},\hat{u}'}\colon V \times (-\epsilon, \epsilon) \to M$ is smooth at $(v, 0)$.

Let us recall the definition of $\hat{u}\#_\rho v$ from [Sch93]. Let $\beta^- : \mathbb{R} \to [0,1]$ be a cut-off function with

$$\beta^-(s) = \begin{cases} 1, & s \leqslant -1,\\ 0, & s \geqslant 0,\end{cases}$$

and $\beta^+(s) = \beta^-(-s)$. We write

$$\beta^\pm_\rho(s) = \beta^\pm(s+\rho), \quad \hat{u}_\rho(s) = \hat{u}(s+\rho)\,.$$

For every $v \in V$ and $\rho \geqslant \rho_o$ large enough we define $w = w(\hat{u}, v, \rho)$ by

$$w = \begin{cases} \hat{u}(s+2\rho), & s \leqslant -\rho - 1,\\ \exp_y\big(\beta^-_\rho \exp_y^{-1} \circ \hat{u}_{2\rho} + \beta^+_\rho \exp_y^{-1} \circ v\big)(s), & |s+\rho| < 1,\\ v(s), & s \geqslant -\rho + 1\,.\end{cases}$$

In particular, $w(\rho) = y$. One can find a $\rho_V > 0$ and a bundle $\pi\colon L^\perp \to V \times [\rho_V, \infty)$ with $L^\perp_{(v,\rho)} \subset C^\infty(w^*TM)$ such that there exists a unique section $\gamma\colon V \times [\rho_V, \infty) \to L^\perp$ providing

$$(\hat{u}\#_\rho v)(s) = \exp_{w(s)}(\gamma(v,\rho)(s)), \quad \hat{u}\#_\rho v \in W^u(x)\,.$$

The bundle $L^\perp$ can be completed fiberwise in terms of a Sobolev space yielding a smooth bundle such that γ is a smooth section. (Details can be found in [Sch93].) Moreover, there is an exponential estimate for the correction term $\gamma(v,\rho)$ between $w(v,\rho)$ and $\hat{u}\#_\rho v$. Namely, there exists a $\sigma > 0$ such that

$$\sup_{s\in\mathbb{R}} |\gamma(v,\rho)(s)| \leqslant c\,\mathrm{e}^{-\sigma\rho} \tag{4.4}$$

for some $c > 0$ uniformly for $v \in V$. Moreover, also the covariant derivatives of γ with respect to v and ρ satisfy such an exponential estimate as $\rho \to \infty$. This is due to the fact that $w(v,\rho) \rightharpoonup (\hat{u}, v)$ as $\rho \to \infty$ and that the gradient flow trajectories $\hat{u}$ and v converge exponentially fast towards y,

$$d(\hat{u}(s), y),\, d(v(-s), y) \leqslant c\, \mathrm{e}^{-\sigma s} \quad \text{ as } s \to \infty\,.$$

The construction of $L^\perp$ and γ in [Sch93] is such that $L^\perp \to V \times [\rho_V, \infty)$ and $L^\perp \to V' \times [\rho_{V'}, \infty)$ coincide over $V \cap V' \times [\max(\rho_V, \rho_{V'}), \infty)$. One obtains a unique smooth gluing map

$$\#_{\hat{u}}^{V \cup V'} \colon (V \cup V') \times [\max(\rho_V, \rho_{V'}), \infty) \to W^u(x)$$

extending $\#^V$ and $\#^{V'}$ and assertion (A) follows.

Let us consider now the coordinate chart $\phi_V(v,t) = \#^V_{\hat{u},\hat{u}'}(v,t)$ with the exponential estimate for the correction term $\gamma(v, \pm\frac{1}{t})$,

$$\|\nabla^\alpha \gamma(v, \pm\tfrac{1}{t})\|_\infty \leqslant c_\alpha\, \mathrm{e}^{-|\frac{1}{t}|} \tag{4.5}$$

where ∇^α are the covariant derivatives of the section γ with respect to the variables v and ρ. We obtain for the evaluation map $E\colon Z(a) \to M$,

$$\begin{aligned} E \circ \phi_V(v,t) &= \begin{cases} E_x(\hat{u}' \#_{(-\frac{1}{t})} v), & t < 0, \\ E_y(v), & t = 0, \\ E_x(\hat{u} \#_{\frac{1}{t}} v), & t > 0, \end{cases} \\ &= \begin{cases} \exp_{v(0)} \big(\gamma(\hat{u}', v, -\frac{1}{t})(0)\big), & t < 0, \\ v(0), & t = 0, \\ \exp_{v(0)} \big(\gamma(\hat{u}, v, \frac{1}{t})(0)\big), & t > 0\,. \end{cases} \end{aligned}$$

Thus, the smoothness of $E \circ \phi_V$ follows from (4.5) and the standard identities for the covariant derivatives of $\exp\colon TM \to M$ at $0_p \in T_pM$. $\square$

The next step is to analyze the map $E\colon Z(a) \to M$ with respect to the end of $Z(a)$, because in general $Z(a)$ is not compact. If it is compact, we immediately obtain the well-defined integral homology class $E_*([Z(a)]) \in H_k(M;\mathbb{Z})$ associated to the Morse cycle a of degree k.

Lemma 4.5. *The evaluation map $E\colon Z(a) \to M$ associated to a Morse cycle $a \in Z_k(f,g)$ is a k-dimensional pseudo-cycle.*

Proof. Consider a point $p \in M$ such that

$$p \in E(Z(a)^\infty) = \bigcap_{K \underset{\text{cpt.}}{\subset} Z(a)} \overline{E(Z(a) \setminus K)}\,.$$

That is, there exists a sequence $(\gamma_n) \subset Z(a)$ such that $\gamma_n(0) \to p$ in M but (γ_n) contains no convergent subsequence in $Z(a)$. We can assume that every γ_n corresponds to an element in $W^u(x)$ for some $x \in \mathrm{Crit}_k f$ such that $a_x \neq 0$ for $a = \sum_x a_x x$. The compactness result for the space of negative gradient flow trajectories provides a convergent subsequence

$$\gamma_{n_k} \xrightarrow{C^\infty_{\text{loc}}} \gamma \in W^u(z),$$

with $\mu(z) \leqslant \mu(x)$. Since (γ_{n_k}) does not converge in

$$Z(a) = \amalg_x a_x \overline{W}^u(x) / \sim\,,$$

we obtain $\mu(z) \leqslant \mu(x) - 2$. This shows that $E(Z(a)^\infty)$ is covered by the images of the evaluation maps

$$E(Z(a)^\infty) \subset \bigcup_{\mu(z) \leqslant k-2} \operatorname{im} E_z \subset M\,. \tag{4.6}$$

Thus, $E\colon Z(a) \to M$ is a pseudo-cycle. □

Given a Morse-cycle $a \in Z_k(f,g)$, we denote the by Lemma 4.5 and Theorem 3.3 uniquely determined homology class by

$$[a] \in H_k(M;\mathbb{Z})\,.$$

Moreover, the map $a \mapsto [a]$ is linear by construction.

Theorem 4.6. *If $a \in Z_k(f,g)$ is a boundary, i.e. $a = \partial b$ for some $b \in C_{k+1}(f,g)$, then $[a] = 0$. That is, the homomorphism*

$$\Phi_{f,g}\colon H_k(C_*(f,g),\partial) \to H_k(M;\mathbb{Z}), \quad \{a\} \mapsto [a] \tag{4.7}$$

is well-defined.

Proof. Consider $a = \sum_x a_x x \in Z_k(f,g)$ and $b = \sum_z b_z z \in C_{k+1}(f,g)$ such that $\partial b = a$. Similar to the construction of $Z(a)$ we now set

$$W = \amalg_{z \in \mathrm{Crit}_{k+1} f} b_z \cdot \overline{W}^u(z), \quad \overline{W}^u(z) = W^u(z) \cup \bigcup_{x \in \mathrm{Crit}_k f} \widehat{M}_{z,x} \times W^u(x)\,,$$

as in (4.1). We can obtain the boundary of the manifold $\overline{W}^u(z)$ as

$$\partial\overline{W}^u(z) = \amalg_{x \in \mathrm{Crit}_k f} n(z,x) \cdot \overline{W}^u(x)\,,$$

such that by setting

$$U = \amalg_{\mu(x)=k} a_x \cdot W^u(x)$$

we obtain the smooth $(k+1)$-dimensional manifold W with boundary $\partial W = U$. Note that $U \subset Z(a)$ is a k-dimensional open submanifold with

$$E(Z(a)^\infty) \subset E(U^\infty)$$

and, analogously to (4.6), $E(U^\infty)$ and $E(W^\infty)$ are covered by the at most $(k-1)$-dimensional submanifolds

$$E(W^\infty) \cup E(U^\infty) \subset \bigcup_{\mu(y) \leqslant k-1} \operatorname{im} E_y\,.$$

Altogether, we have $E(W^\infty) \cap E(U) = \emptyset$ and $H_k(E(W^\infty);\mathbb{Z}) = 0$ so that Theorem 3.1 (b) is applicable. □

In order to obtain a homomorphism $\Phi\colon H_*^{\mathrm{Morse}}(M) \to H_*^{\mathrm{sing}}(M)$ we have to show that the linear maps $\Phi_{f,g}$ are compatible with the canonical isomorphisms from Theorem 2.2

$$\Phi_{10}\colon H_*(f_0,g_0) \xrightarrow{\cong} H_*(f_1,g_1)$$

from Theorem 2.2. For this purpose we first present an alternative construction of a pseudo-cycle associated to a (f,g)-Morse cycle representing the same singular homology class.

Let us consider a smooth 1-parameter family (f_s,g_s) of functions and Riemannian metrics with $-\infty < s \leqslant 0$ such that for some $R > 0$

$$(f_s,g_s) = (f,g), \quad \text{for all } s < -R\,.$$

The pair (f,g) is Morse-Smale as above. We now redefine for $x \in \operatorname{Crit} f$

$$W^u(x) = \{\, \gamma\colon (-\infty,0] \to M \mid \dot\gamma(s) + \nabla_{g_s} f_s(\gamma(s)) = 0,\ \gamma(-\infty) = x \,\}\,.$$

All statements about the prior unstable manifolds remain valid for this non-autonomous flow. Observe that we have the following weak compactness result: Given any sequence $(\gamma_n) \subset W^u(x)$ which contains no convergent subsequence, there exists a subsequence (n_k) and a reparametrization sequence $\tau_k \to -\infty$ such that

$$\gamma_{n_k} \xrightarrow{C^\infty_{\mathrm{loc}}} \gamma \in W^u(y) \quad \text{and} \quad \tau_k * \gamma_{n_k} \xrightarrow{C^\infty_{\mathrm{loc}}} u \in M_{x',y'}(f,g)\,.$$

for some $x, x', y', y \in \operatorname{Crit} f$ with $\mu(x) \geqslant \mu(x') > \mu(y') \geqslant \mu(y)$. Again, in general, γ_{n_k} can converge weakly towards a multiply broken trajectory. If $\mu(y) = \mu(x) - 1$ we have $x' = x$ and $y' = y$.

Carrying out the same constructions as before, now based on the deformed unstable manifolds associated to (f_s, g_s), we obtain pseudo-cycles

$$\tilde E\colon \tilde Z(a) \to M$$

associated to Morse-cycles $a \in Z_k(f,g)$ thus leading to a homomorphism

$$\widetilde\Phi_{f,g}\colon H_*(C_*(f,g),\partial) \to H_*(M;\mathbb{Z})\,. \tag{4.8}$$

Lemma 4.7. *The homomorphisms $\Phi_{f,g}$ and $\widetilde\Phi_{f,g}$ are identical.*

Proof. We have to show that the pseudo-cycles $E\colon Z(a) \to M$ and $\tilde E\colon \tilde Z(a) \to M$ can be related by a suitable pseudo-cycle cobordism such that Theorem 3.1 (b) applies.

Since asymptotically constant families $(f_s, g_s)_{s \in (-\infty,0]}$ as above form a convex set we can consider the continuous path

$$(f_s, g_s)_\lambda = (1-\lambda)(f,g) + \lambda(f_s, g_s), \quad \lambda \in I = [0,1]\,.$$

Given $x \in \operatorname{Crit} f$, the space

$$W^u_I(x) = \{\, (\lambda,\gamma) \mid \lambda \in I,\ \gamma \in W^u_\lambda(x) \,\}$$

is a smooth manifold of dimension $\mu(x)+1$, where $W^u_\lambda(x)$ is the unstable manifold of x associated to the pair $(f_s,g_s)_\lambda$. Its boundary is the disjoint union of $W^u_1(x)$ and $W^u_0(x)^*$, i.e. the latter with reversed orientation. Analyzing the non-compactness of $W^u_I(x)$, we consider a sequence (λ_n, γ_n) which contains no convergent subsequence. There exists a subsequence (n_k) such that $\lambda_{n_k} \to \lambda$ and either

$$\gamma_{n_k} \rightharpoonup (u,\gamma) \in \widehat M_{x,y}(f,g) \times W^u_\lambda(x)\,,$$

for $\mu(y) = \mu(x) - 1$, or γ_{n_k} converges in C^∞_{loc} towards a $\gamma \in W^u_\lambda(z)$ with $\mu(z) \leqslant \mu(x) - 2$.

Moreover, we can prove a λ-parameterized version of the gluing result in Lemma 4.1 yielding a gluing map

$$\#^V\colon \widehat M_{x,y} \times V \times [\rho_V, \infty) \to W^u_I(x)$$

for every relatively compact, open subset $V \subset W^u_I(y)$ and $\mu(y) = \mu(x) - 1$. This allows us to build $\overline{W}^u_I(x)$ as in (4.1) and to construct a smooth manifold $Z_I(a)$ together with a smooth map

$$E\colon Z_I(a) \to M, \quad \partial Z_I(a) = \tilde Z(a) - Z(a),$$

which extends the given maps $\tilde{E}$ and E on the boundary. Since

$$E(Z_I(a)^\infty) \subset \bigcup_{\mu(z)\leqslant\mu(x)-2} \operatorname{im} E_z$$

for $E_z\colon W_I^u(z) \to M$ with $\dim W_I^u(z) \leqslant \mu(x)-1$, we meet the conditions of Theorem 3.1 (b). □

In view of (2.2), we now show that the pseudo-cycle homomorphisms $\Phi_i = \Phi_{(f^i,g^i)}$ are compatible with the canonical isomorphisms Φ_{ij},

Lemma 4.8. *The homomorphisms $\Phi_i\colon H_*(f^i,g^i) \to H_*(M;\mathbb{Z})$ are compatible with (Φ_{ij}), that is,*

$$\Phi_1 \circ \Phi_{10} = \Phi_0$$

for all Morse-Smale pairs (f^0,g^0) and (f^1,g^1).

Proof. We have to compare the pseudo-cycles

$$E^0\colon Z^0(a) \to M \quad \text{and} \quad E^1\colon Z^1(\Phi_{10}(a)) \to M$$

for any Morse cycle $a \in Z_k(f^0,g^0)$. That is, we have to show that the E^i can be extended to a suitable cobordism $E\colon W \to M$ such that, again, Theorem 3.1 (b) applies.

Let us consider the space similar to $W_I^u(x)$ in the proof of Lemma 4.7,

$$W_{\mathbb{R}_+}(x) = \{\, (\lambda,\gamma) \mid \lambda \in [0,\infty),\ \gamma \in W^u(x; f_{s+\lambda}, g_{s+\lambda})\,\}$$

for $x \in \operatorname{Crit} f^0$. (If $[0,\infty)$ is replaced by a compact interval we are in the situation of Lemma 4.7.) Now we have to deal with additional non-compactness for $\lambda_n \to \infty$. Let $(\lambda_n,\gamma_n) \subset W_{\mathbb{R}_+}(x)$ be such that $\lambda_n \to \infty$. Then, there exists a subsequence (n_k) such that

$$\gamma_{n_k} \xrightarrow{C^\infty_{\text{loc}}} \gamma \in W^u(x'; f^1, g^1)$$

for some $x' \in \operatorname{Crit} f^1$. Necessarily, $\mu(x') \leqslant \mu(x)$. If both critical points x and x' have equal Morse index then, up to choosing a subsequence,

$$(-\lambda_{n_k}) * \gamma_{n_k} \xrightarrow{C^\infty_{\text{loc}}} u \in M_{x,x'}(f_s,g_s)\,.$$

In that case we denote this weak convergence again by

$$(\lambda_{n_k},\gamma_{n_k}) \rightharpoonup (u,\gamma)\,.$$

For the converse, we have a gluing theorem analogous to Lemma 4.1:

Let $\mu(x) = \mu(x')$. Given $V \subset W^u(x')$, an open and relatively compact subset, there exists a $\lambda_V > 0$ and a smooth map

$$\#^V\colon M_{x,x'}(f_s,g_s) \times V \times [\lambda_V,\infty) \to W_{\mathbb{R}_+}(x)\,,$$

such that the corresponding properties (a)–(d) as in Lemma 4.1 hold true. Extending the construction from the proof of Lemma 4.7 based on the λ-parametrized gluing, we now glue in boundary manifolds to $W_{\mathbb{R}_+}$ such that

$$\begin{aligned}\overline{W}_{\mathbb{R}_+}(x) = W_{\mathbb{R}_+}(x) &\cup \bigcup_{\mu(y)=\mu(x)-1} \big(M_{x,y}(f^0,g^0) \times W_{\mathbb{R}_+}(y)\big) \\ &\cup \bigcup_{\mu(x')=\mu(x)} \big(M_{x,x'}(f_s,g_s) \times W^u(x'; f^1,g^1)\big)\,.\end{aligned}$$

Note that it is not necessary to glue in the codimension-2 manifolds $M_{x,y}(f^0,g^0)\times M_{y,y'}(f_s,g_s)\times W^u(y';f^1,g^1)$ for $\mu(y')=\mu(x)-1$. Building the quotient manifold

$$\bar{Z}(a)=\amalg_{x\in\operatorname{Crit}_k f^0} a_x\cdot\overline{W}_{\mathbb{R}_+}(x)/\sim$$

analogously as above, we obtain a smooth $(k+1)$-dimensional manifold with boundary

$$\partial\bar{Z}(a)=Z^o(\Phi_{10}(a))-Z(a)$$

where $Z^o(\Phi_{10}(a))$ is the open subset

$$\bigcup_{x\in\operatorname{Crit} f^0} a_x\cdot M_{x,x'}\times W^u(x')\subset Z(\Phi_{10}(a))$$

with complementary strata of codimension at least 1. The evaluation maps $E(\gamma)=\gamma(0)$ extend from the boundary manifolds to $\bar{Z}(a)$ and it is straightforward to verify that the conditions for Theorem 3.1 (b) are satisfied. $\square$

Summing up, we obtain the well-defined homomorphism

$$\Phi\colon H_*^{\text{Morse}}(M;\mathbb{Z})\to H_*^{\text{sing}}(M;\mathbb{Z})\,. \tag{4.9}$$

4.1.1. *Remarks on compatibility with other equivalences for Morse homology.* In view of the axiomatic approach to Morse homology adopted in [Sch93], it is straightforward, based on Lemma 4.8, to verify that the homomorphism Φ is natural. This means it respects functoriality with respect to closed embeddings, w.r.t. changes of Morse functions, and it is compatible with the relative version of Morse homology. Thus, we can refer to the uniqueness result from [Sch93], mentioned in 2.2, in order to conclude that Φ is in fact the unique, natural isomorphism between Morse homology and singular homology.

Let us also remark that in case of a self-indexing Morse function f, i.e.

$$\mu(x)=f(x),\ \forall x\in\operatorname{Crit} f,$$

we have the obvious isomorphism

$$\Gamma_k\colon C_k(f)\xrightarrow{\cong}H_k^{\text{sing}}(M^k,M^{k-1};\mathbb{Z}), \tag{4.10}$$

for $M^a=\{\,p\in M\mid f(p)\leqslant a\,\}$, $a\in\mathbb{R}$. A classical proof for the equivalence of Morse homology and singular homology[6] is to show

$$\Gamma_{k-1}\circ\partial(f,g)=\partial_*\circ\Gamma_k \tag{4.11}$$

for the boundary operator in the long exact sequence associated to the decomposition $(M^k,M^{k-1})_{k=0,\dots,n}$,

$$H_k(M^k,M^{k-1})\xrightarrow{\partial_*}H_{k-1}(M^{k-1},M^{k-2})\,. \tag{4.12}$$

Obviously, we have for $j\colon H_k(M^k)\to H_k(M^k,M^{k-1})$,

$$j\circ\Phi_{fg}(a)=\Gamma(a),\ \forall a\in Z_k(f,g) \tag{4.13}$$

so that the induced isomorphism $\Gamma\colon H_*(f,g)\xrightarrow{\cong}H_*^{\text{sing}}(M;\mathbb{Z})$ and $\Phi_{f,g}$ are identical.

[6] used in [Mil65]

4.2. **The Inverse Homomorphism.** Although it is already clear that the homomorphism

$$\Phi\colon H_*^{\mathrm{Morse}}(M;\mathbb{Z}) \to H_*^{\mathrm{sing}}(M;\mathbb{Z})$$

is a natural isomorphism, let us nevertheless construct its inverse $\Psi = \Phi^{-1}$ *explicitly* along the same lines as used for Φ.

The main idea is to define an intersection number for pseudo-cycles and stable manifolds $W^s(x)$ for a generic Morse-Smale pair (f,g). Recall from [Sch93] the construction of a Banach manifold $\mathcal{G}$ of smooth Riemannian metrics, L^2-dense in the space of all smooth Riemannian metrics. We consider a metric g to be generic with respect to a certain property, if we can find a residual set $\mathcal{R} \subseteq \mathcal{G}$ of metrics with that property. The first step is

Theorem 4.9. *Let $\chi\colon V^k \to M$ be a smooth map of a k-dimensional manifold into M and f a Morse function on M such that $\chi(V) \cap \operatorname{Crit} f = \emptyset$ if $k < n$ or $\operatorname{rank} D\chi(p) = n$ for all $p \in \chi^{-1}(\operatorname{Crit} f)$ if $k = n$. Then there exists a residual set $\mathcal{R} \subseteq \mathcal{G}$ such that*

$$\mathcal{M}_{\chi;x}(f,g) = \{\, (p,\gamma) \in V \times W^s(x) \mid \dot\gamma + \nabla_g f(\gamma) = 0,\ \gamma(0) = \chi(p) \,\}$$

is a smooth manifold of dimension

$$\dim \mathcal{M}_{\chi;x}(f,g) = k - \mu(x)$$

for all $x \in \operatorname{Crit} f$ and $g \in \mathcal{R}$. In particular, it is empty if $k < \mu(x)$.

Also, as will be clear from the proof, if V^k, M and $W^s(x)$ are oriented, the intersection manifold $\mathcal{M}_{\chi;x}(f,g)$ inherits a well-defined orientation which is a number $\pm 1 \in \mathbb{Z}_2$ if $k = \mu(x)$.

Proof. The main ingredient of this transversality theorem is the following

Lemma 4.10. *The universal stable manifold*

$$W^s_{univ}\{\, (\gamma,g) \in C^\infty([0,\infty),M) \times \mathcal{G} \mid \dot\gamma + \nabla_g f(\gamma) = 0,\ \gamma(+\infty) = x \,\}$$

for $x \in \operatorname{Crit} f$, f a Morse function, admits a submersion

$$E\colon W^s_{univ}(x) \to M, \quad E(\gamma,g) = \gamma(0),$$

away from the critical point $\gamma \equiv x$. It is also a submersion everywhere if $\mu(x) = 0$.

Proof. Let us recall some analytic constructions from [Sch93]. The space

$$\mathcal{H}^{1,2}_x = H^{1,2}_x([0,\infty),M)$$

is the $H^{1,2}$-Sobolev completion of the space of smooth curves $\gamma\colon [0,\infty) \to M$ with sufficiently fast convergence toward $x \in M$ as $t \to \infty$. It is in fact a Hilbert manifold. The tangent space to the Banach manifold of smooth Riemannian metrics on M is

$$T_g\mathcal{G} = \{\, h \in C^\infty_\epsilon(\operatorname{End}(TM)) \mid h \,\text{symmetric w.r.t. } g_o \,\}$$

for some fixed Riemannian metric g_o. The function space C^∞_ϵ is an L^2-dense subspace of C^∞ with a Banach space norm. Let us now consider the smooth map

$$F\colon \mathcal{H}^{1,2}_x \times \mathcal{G} \to L^2(\mathcal{H}^{1,2*}_x TM),$$
$$F(\gamma,g) = \dot\gamma + (\nabla_g f) \circ \gamma,$$

where the right hand side space is a Banach space bundle over the manifold $\mathcal{H}^{1,2}_x$ with fiber $L^2(\gamma^*TM)$ of L^2-vector fields along the curve γ. Choosing a Riemannian connection ∇ on TM we obtain the linearization of F as

$$\begin{aligned} DF(\gamma,g)(\xi,h) &= DF_1(\gamma,g)(\xi) + DF_2(\gamma,g)(h), \\ DF_1(\gamma,g)(\xi) &= \nabla_t\xi + (\nabla_\xi\nabla_g f)\circ\gamma, \\ DF_2(\gamma,g)(h) &= h(\gamma)\cdot\nabla_g f(\gamma)\,. \end{aligned}$$

Observe that $h(\gamma)$ is an endomorphism of the pull-back bundle γ^*TM. Hence, any variation of $h(\gamma(t))$ as a function of time t can be achieved through a variation of h over M if γ is injective. Altogether we obtain the tangent space of the universal stable manifold as

$$T_{(\gamma,g)}W^s_{\mathrm{univ}}(x) = \{\,(\xi,h)\,|\,DF(\gamma,g)(\xi,h)=0\,\}, \tag{4.14}$$

because 0 is a regular value for F, as it will be clear below. Given $\gamma(0)=p\in M$ and $(\gamma,g)\in W^s_{\mathrm{univ}}(x)$ we have to show that for each $v\in T_pM$ there exist $(\xi,h)\in T_{(\gamma,g)}W^s_{\mathrm{univ}}(x)$ such that $\xi(0)=v$, if either

(a) $\gamma(0)\notin \mathrm{Crit}\,f$, i.e. $\gamma(0)\neq x$, or
(b) $\gamma(0)=x$ and $\mu(x)=0$.

In the latter case (b) we have $\gamma\equiv\mathrm{const}=x$ with

$$DF_1(x,g)(\xi) = \dot\xi + \mathrm{Hess}\,f(x)\cdot\xi, \quad DF_2(x,g)(h)=0,$$

where the Hessian at $x\in\mathrm{Crit}_0\,f$ is positive definite. This implies

$$\ker DF(x,g) = T_xM\times T_g\mathcal{G},$$

and hence the submersion property of E.

In case (a) let us simplify the operator $DF(\gamma,g)$ by using coordinates with respect to an orthonormal parallel frame of γ^*TM. We obtain the operator,

$$\begin{aligned} &D\colon H^{1,2}([0,\infty),\mathbb{R}^n)\times T_g\mathcal{G}\to L^2([0,\infty),\mathbb{R}^n) \\ &D(\xi,h) = \dot\xi + A(t)\xi + h\cdot X, \end{aligned} \tag{4.15}$$

where $A\colon[0,\infty)\to S(n,\mathbb{R})$ is a smooth path in the space of symmetric $n\times n$-matrices with $A(\infty)=\mathrm{Hess}\,f(x)$ and $X\colon[0,\infty)\to\mathbb{R}^n$ with $X(t)\neq 0$ for all $t\in[0,\infty)$. We shall now prove that for all $\eta\in L^2([0,\infty),\mathbb{R}^n)$ and $v\in\mathbb{R}^n$ there exist $\xi\in H^{1,2}([0,\infty),\mathbb{R}^n)$ and $h\in C^\infty_o([0,\infty),S(n))$ such that $D(\xi,h)=\eta$ and $\xi(0)=v$. This concludes the proof of (a) in view of the fact that each such h arises from an $h\in T_g\mathcal{G}$ since γ is injective if $\gamma(0)\neq x$.

Suppose that there exist η and v such that

$$\langle D(\xi,h),\eta\rangle_{L^2} + \langle\xi(0),v\rangle_{\mathbb{R}^n} = 0 \quad \text{for all } \xi,h\,. \tag{4.16}$$

This implies that $\eta\in H^{1,2}([0,\infty),\mathbb{R}^n)$ and $\dot\eta - A^t(t)\eta = 0$ and therefore $\eta\equiv 0$ if $\eta(0)=0$. Moreover, (4.16) implies that $\langle hX,\eta\rangle = 0$ for all h and we have $X(t)\neq 0$. If $\eta(0)\neq 0$ we can find[7] $h(t)$ with support in $[0,\epsilon)$ such that $\langle hX,\eta\rangle\neq 0$ contradicting (4.16). Hence we obtain $\eta\equiv 0$ and by (4.16) $\langle\xi(0),v\rangle_{\mathbb{R}^n}=0$ for all ξ which implies $v=0$. Since the cokernel of D in L^2 is finite-dimensional it follows that D is surjective. □

[7] Compare (2.38) in the proof of Proposition 2.30 in [Sch93]

The proof of Theorem 4.9 now follows from the parameter version of the Sard-Smale theorem. There exists a residual subset $\mathcal{R} \subseteq \mathcal{G}$ such that for each $g \in \mathcal{R}$ the map

$$(\chi, E)\colon V^k \times W^s_g(x) \to M \times M$$

intersects the diagonal $\triangle = \{\,(p,p)\,|\,p \in M\}$ transversely. For such a generic g,

$$\mathcal{M}_{\chi;x}(f,g) = (\chi, E)^{-1}(\triangle)$$

is a smooth manifold of dimension $k + (n-\mu) - n$.

Note that for $k = \mu(x)$ intersections $(p,\gamma) \in \mathcal{M}_{\chi;x}(f,g)$ for a regular g can only occur if $\operatorname{rank} D\chi(p) = k$. Therefore, it is obvious how the solution space $\mathcal{M}_{\chi;x}$ inherits its orientation from an orientation of V^k, M and $W^s(x)$. $\square$

The main consequence of the intersection theorem 4.9 is the compactness result

Corollary 4.11. *For each k-dimensional pseudo-cycle $\chi\colon V^k \to M$ with $\overline{\chi(V)} \cap \operatorname{Crit} f = \emptyset$ if $k < n$, and $\operatorname{rank} D\chi(p) = n$ for all $p \in \chi^{-1}(\operatorname{Crit} f)$ and $\chi(V^\infty) \cap \operatorname{Crit} f = \emptyset$ if $k = n$, there is a residual set of metrics $\mathcal{R}$ such that the intersection set $\mathcal{M}_{\chi;x}(f,g)$ is finite for all $x \in \operatorname{Crit} f$ with $\mu(x) = k$ and $g \in \mathcal{R}$.*

Proof. Consider a sequence $(p_n, \gamma_n) \subseteq \mathcal{M}_{\chi;x}(f,g)$. After choosing a suitable subsequence we have

$$\chi(p_n) \to x_o \in \overline{\chi(V)}, \quad \gamma_n \xrightarrow{C^\infty_{\mathrm{loc}}} \gamma_o \in W^s(x'),\ \mu(x') \geqslant \mu(x),\ \gamma_o(0) = x_o\,. \tag{4.17}$$

In the case that $x_o \in \chi(V^\infty)$, we use that $\chi(V^\infty)$ can be covered by a map $\tilde\chi\colon \tilde V^{k-2} \to M$, so that $(p_o, \gamma_o) \in \mathcal{M}_{\tilde\chi;x'}(f,g)$, $\tilde\chi(p_o) = x_o$. Since the intersection of residual sets is residual it follows from Theorem 4.9 that for a generic g $\mathcal{M}_{\tilde\chi;x'}$ has to be empty by dimensional reasons. Thus $x_o \in \chi(V^\infty)$ can be excluded.

Sharpening the convergence result (4.17) we can deduce weak convergence towards a broken trajectory

$$\gamma_n \rightharpoonup (\gamma_o, u_1, \dots, u_r), \quad \gamma_o \in W^s(x'),\ u_1 \in \mathcal{M}_{x',x_1}, \dots,\ u_r \in \mathcal{M}_{x_{r-1},x}\,. \tag{4.18}$$

For such multiply broken trajectories we must have $\mu(x_{i-1}) > \mu(x_i)$. Hence, if $k = \mu(x)$ we cannot have $x' \neq x$ and $\mathcal{M}_{\chi;x}(f,g)$ must be compact and hence finite. $\square$

Applying the concept of coherent orientations we can now associate to each intersection $(p,\gamma) \in \mathcal{M}_{\chi;x}(f,g)$ a sign $\tau(p,\gamma)$. Given a k-dimensional pseudo-cycle $\chi\colon V^k \to M$ representing a singular cycle $\alpha = \alpha_\chi \in H^{\mathrm{sing}}_k(M)$ with $k < n$ we can find a Morse function f such that $\operatorname{Crit} f \cap \overline{\chi(V)} = \emptyset$. If $k = n$, after possibly homotoping χ to a suitable cobordant pseudo-cycle, we can find a Morse function f such that we have only $p \in \chi^{-1}(\operatorname{Crit} f)$ with $\operatorname{rank} D\chi(p) = n$. We then define in view of Theorem 4.9 for a generic g

$$\Psi(\chi) = \sum_{x \in \operatorname{Crit}_k f} \#_{\mathrm{alg}} \mathcal{M}_{\chi;x}(f,g)\, x\ \in C_k(f,g)\,. \tag{4.19}$$

Corollary 4.12. *The chain $\Psi(\chi) \in C_k(f,g)$ is a Morse-cycle, and given two cobordant pseudo-cycles χ, χ', the associated Morse-cycles are cohomologous, $\Psi(\chi) - \Psi(\chi') = \partial(f,g)b$ for some $b \in C_{k+1}(f,g)$.*

Proof. Computing

$$\partial(f,g)\sum_{x\in\mathrm{Crit}_k f}\#_{\mathrm{alg}}\mathcal{M}_{\chi;x}(f,g)\,x=\sum_{\mu(y)=k-1}n(\chi;y)\,y,$$

$$n(\chi;y)=\sum_{\mu(x)=k}\#_{\mathrm{alg}}\mathcal{M}_{\chi;x}(f,g)\#_{\mathrm{alg}}\mathcal{M}_{x;y}(f,g),$$

we have to show that

$$n(\chi;y)=0\,. \tag{4.20}$$

This follows readily from the 1-dimensional compactness result for $\mathcal{M}_{\chi;y}$ analogous to (4.17) and (4.18) and the corresponding gluing operation. Namely, since $\dim\chi-\mu(y)=1$, non-compactness of $\mathcal{M}_{\chi;y}(f,g)$ for generic g can only occur in terms of simply broken trajectories in the limit. But exactly as for the proof of the fundamental fact $\partial(f,g)^2=0$, the corresponding gluing result completely analogous to Lemma 4.1 shows that the oriented number of boundary components of $\mathcal{M}_{\chi;y}(f,g)$ equals $n(\chi;y)$ and has to vanish since each component of $\mathcal{M}_{\chi;y}$ is diffeomorphic to an interval. This proves (4.20).

Given a pseudo-cycle cobordism $F\colon W^{k+1}\to M$ in the sense of Theorem 3.1 (b), i.e. $\partial F=\chi-\chi'$, we can define the 1-dimensional manifold $\mathcal{M}_{F;x}(f,g)$ for $x\in\mathrm{Crit}_k f$, and generic g. The same compactness-gluing argument as before shows

$$\partial(f,g)\sum_{x\in\mathrm{Crit}_k f}\#_{\mathrm{alg}}\mathcal{M}_{F;x}(f,g)\,x=\Psi(\chi)-\Psi(\chi')\,. \tag{4.21}$$

□

The thus well-defined homomorphism

$$\Psi_{f,g}\colon H_*^{\mathrm{sing}}(M;\mathbb{Z})\to H_*(f,g)$$

is compatible with the canonical isomorphism

$$\Phi_{10}\colon H_*(f^0,g^0)\to H_*(f^1,g^1)\,.$$

We have

Corollary 4.13. *Considering the isomorphism Φ_{10} for generic Morse-Smale pairs, it holds*

$$\Phi_{10}\circ\Psi_{f^0,g^0}=\Psi_{f^1,g^1},$$

and we have for the well-defined homomorphism

$$\Psi\colon H_*^{sing}(M;\mathbb{Z})\to H_*^{Morse}(M;\mathbb{Z})$$

the identity $\Psi\circ\Phi=\mathrm{id}_{H^{Morse}}$.

Proof. The proof of the compatibility $\Phi_{10}\circ\Psi_{f^0,g^0}=\Psi_{f^1,g^1}$ can be carried out exactly analogous to that for Lemma 4.8 using the argument from the proofs of Corollaries 4.11 and 4.12.

Consider now the k-dimensional pseudo-cycle $E\colon Z(a)\to M$ associated to a Morse-cycle $a\in Z_k(f^0,g^0)$. Let $y\in\mathrm{Crit}_k f^1$. Then in view of Theorem 4.9 for a Morse-Smale pair (f^1,g^1) with generic g^1 we have intersections $(p,\gamma)\in\mathcal{M}_{E;y}(f^1,g^1)$ only for $p\in Z(a)$ on the k-dimensional strata which are exactly the unstable manifolds $W^u(x,f^0,g^0)$ in $a=\sum_{x\in\mathrm{Crit}_k f^0}a_x\,x$. We therefore have

$$\mathcal{M}_{E;y}(f^1,g^1)=\{\,(\gamma^-,\gamma^+)\in W^u(x,f^0,g^0)\times W^s(y;f^1,g^1)\,|\,\gamma^-(0)=\gamma^+(0)\,\}\,.$$

Using a homotopy operator as before we can show easily that this is homologically equivalent to the definition of the operator $\Phi_{10}\colon C_*(f^0,g^0)\to C_*(f^1,g^1)$, i.e.

$$\Psi_{f^1,g^1}\circ\Phi_{f^0,g^0}=\Phi_{10}\colon H_*(f^0,g^0)\xrightarrow{\cong}H_*(f^1,g^1)\,. \tag{4.22}$$

This proves $\Psi\circ\Phi=\mathrm{id}_{H^{\mathrm{Morse}}}$. □

Using the fact that $H_*^{\mathrm{Morse}}(M)\cong H_*^{\mathrm{sing}}(M)$, it follows immediately that the left-inverse Ψ is the inverse of Φ. Thus we have explicit constructions of both isomorphisms in terms of Morse-pseudo-cycle equivalences.

References

[Con78] C.C. Conley, *Isolated invariant sets and the Morse index*, CBMS Reg. Conf. Series in Math., vol. 38, A.M.S., Providence, R.I., 1978.

[DK90] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1990.

[Flo89] A. Floer, *Witten's complex and infinite dimensional Morse theory*, J. Diff. Geom. **30** (1989), 207–221.

[Fra79] J.M. Franks, *Morse-smale flows and homotopy theory*, Topology **18** (1979), 199–215.

[Mas78] W. S. Massey, *Homology and cohomology theory*, Monographs and textbooks in pure and applied mathematics, vol. 46, Marcel Dekker, New York, 1978.

[Mil65] J.W. Milnor, *Lectures on the h-cobordism theorem*, Math. Notes, Princeton Univ. Press, Princeton, 1965.

[MS94] D. McDuff and D. Salamon, *J-holomorphic curves and quantum cohomology*, University Lecture Series, vol. 6, A.M.S., Providence, R.I., 1994.

[PSS96] S. Piunikhin, D. Salamon, and M. Schwarz, *Symplectic Floer-Donaldson theory and quantum cohomology*, Contact and Symplectic Geometry (Cambridge 1994) (C.B. Thomas, ed.), Publ. Newton Inst., vol. 8, Cambridge University Press, 1996, pp. 171–200.

[Sch] M. Schwarz, *An explicit isomorphism between Floer homology and quantum cohomology*, in preparation.

[Sch93] M. Schwarz, *Morse homology*, Progress in Mathematics, vol. 111, Birkhäuser, Basel, 1993.

[Sch98] M. Schwarz, *A quantum cup-length estimate for symplectic fixed points*, Invent. math. **133** (1998), 353–397.

[Sei97] P. Seidel, π_1 *of symplectic automorphism groups and invertibles in quantum homology rings*, Geom. funct. anal. **7** (1997), 1046–1095.

[Wit82] E. Witten, *Supersymmmetry and Morse theory*, J. Diff. Geom. **17** (1982), 661–692.

Department of Mathematics, University of Chicago, Chicago, IL 60637
E-mail address: schwarz@math.uchicago.edu